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About the uncertainty of modal parameters estimated in Operational Modal Analysis

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ABSTRACT

A significant challenge in estimating modal parameters from output-only data is the computation of the uncertainty. Despite its importance, limited theoretical work has been dedicated to understanding this issue. This study presents key findings derived from the application of random vibration theory to the estimation of damping ratios and natural frequencies. The results offer new insights into the factors affecting the accuracy of these estimates and emphasize the need for further investigation to enhance the precision of modal parameter identification in dynamic systems.

Keywords: Operational Modal Analysis, uncertainty, random vibrations, Delta method

1. INTRODUCTION

Operational Modeal Analysis consists of the estimation of modal parameters from the vibrations recorded in the structures using sensors. An important aspect of this method is the analysis of the uncertaintity of the estimated modal parameters. There are some works in the technical literature about this issue. For example, [1] proposed a method to compute the variance of the parameters estimated in the frequency domain; [2] presented a method to find the variance matrix in the case of the Stochastic Subspace Identification algoritm, maybe the most important algorithm in OMA; this method has been improved in subsequent works, like [3] and [4]; another important direction of research is to use the Bayesian approach, like in [5].

The porpose of this work is to find the uncertainty of the modal parameters in the context of random vibrations. The idea is to identify the main factors that influence the uncertainty of the estimated parameters regardless of the estimation method that has been used.

2. RANDOM VIBRATIONS OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM

The equation modeling the vibrations of a single-degree of freedom system is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{1}$$

where m is the mass of the system [kg], c is the damping [N·s/m], k is the system stiffness [N/m], $F(t)$ is the external force and $x(t)$ is the position of the system at time t. This equation is usually written as

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \frac{1}{m}F(t) \tag{2}$$

where ω_n is the natural frequency of vibration and ζ is the damping ratio:

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2m\omega_n} \tag{3}$$

In Operational Modal Analysis the system force $F(t)$ is unknown so the option is to model it as a random process: in this work, we model this force as a white noise process. Therefore, the autocorrelation function is

$$R_{FF}(\tau) = E[F(t)F(t+\tau)] = 2\pi S_0\delta(\tau), \quad -\infty \leq \tau \leq \infty \tag{4}$$

where $E[\,]$ is the expected value, $\delta()$ is the Dirac delta function. And the power spectral density function is

$$S_{FF}(\omega) = \int_{-\infty}^{\infty} R_{FF}(\tau)e^{-i\omega\tau}d\tau = S_0, \quad -\infty \leq \omega \leq \infty \tag{5}$$

Under this hypothesis, we have:

Property 1. *For the dynamic system given by equation* (1)*, when the driven force $F(t)$ is a random process with $E[F(t)] = 0$ and power spectral density funtion $S_{FF}(\omega) = S_0$, we have:*

$$Var[x(t)] = \frac{S_0\pi}{2\zeta m^2\omega_n^3} \tag{6}$$

$$Var[\dot{x}(t)] = \frac{S_0\pi}{2m^2\zeta\omega_n} \tag{7}$$

$$Var[\ddot{x}(t)] = \frac{S_0\pi\omega_n}{2m^2\zeta}(1-4\zeta^2) \tag{8}$$

$$Cov[x(t), \dot{x}(t)] = 0 \tag{9}$$

$$Cov[x(t), \ddot{x}(t)] = -\frac{S_0\pi}{2m^2\zeta\omega_n} \tag{10}$$

$$Cov[\dot{x}(t), \ddot{x}(t)] = -\frac{S_0\pi}{m^2} \tag{11}$$

The proof of this property can be obtained using [7], [8] and [9].

3. MODAL PARAMETER ESTIMATION FROM OUTPUT-ONLY DATA

3.1. Estimators of the modal parameters

Let us consider a single-degree-of-freedom system where we measure the acceleration, velocity and position for n different time steps: $\ddot{x}_k, \dot{x}_k, x_k,\ k = 1, 2, \ldots, n$. From these measurements, we can estimate the natural frequency and damping ratio of the system using a linear regression model:

$$\ddot{x}_k = \beta_1 x_k + \beta_2 \dot{x}_k + u_k, \quad u_k \sim N(0, \sigma^2), \quad k = 1, 2, \ldots, n. \tag{12}$$

The equation of the estimated model is

$$\ddot{x}_k = \hat{\beta}_1 x_k + \hat{\beta}_2 \dot{x}_k + e_k, \quad k = 1, 2, \ldots, n. \tag{13}$$

where e_k are the residuals. Comparing equations (2) and (13), we can estimate the natural frequency and damping ratio by mean of:

$$\hat{\beta}_1 = -\hat{\omega}_n^2 \Rightarrow \hat{\omega}_n = \sqrt{-\hat{\beta}_1} \tag{14}$$

$$\hat{\beta}_2 = -2\hat{\omega}_n\hat{\zeta} \Rightarrow \hat{\zeta} = -\frac{\hat{\beta}_2}{2\hat{\omega}_n} \tag{15}$$

One important property of the residuals is:

$$\sum_{k=1}^{n} e_k = 0 \tag{16}$$

Taking into account this property, we can write:

$$\bar{\ddot{x}} = \hat{\beta}_1 \bar{x} + \hat{\beta}_2 \bar{\dot{x}} \tag{17}$$

where $\bar{\bullet}$ stands for the sample mean value:

$$\bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k, \quad \bar{\dot{x}} = \frac{1}{n}\sum_{k=1}^{n} \dot{x}_k, \quad \bar{\ddot{x}} = \frac{1}{n}\sum_{k=1}^{n} \ddot{x}_k. \tag{18}$$

Using equations (13) and (17) we can write:

$$(\ddot{x}_k - \bar{\ddot{x}}) = \hat{\beta}_1(x_k - \bar{x}) + \hat{\beta}_2(\dot{x}_k - \bar{\dot{x}}) + e_k, \quad k = 1, 2, \ldots, n. \tag{19}$$

These equations can be written in matrix form as:

$$\mathbf{Y} = \mathbf{X}\hat{\boldsymbol{\beta}} + \mathbf{e} \tag{20}$$

where

$$\mathbf{Y} = \begin{bmatrix} \ddot{x}_1 - \bar{\ddot{x}} \\ \ddot{x}_2 - \bar{\ddot{x}} \\ \cdots \\ \ddot{x}_n - \bar{\ddot{x}} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 - \bar{x} & \dot{x}_1 - \bar{\dot{x}} \\ x_2 - \bar{x} & \dot{x}_2 - \bar{\dot{x}} \\ \cdots & \cdots \\ x_n - \bar{x} & \dot{x}_n - \bar{\dot{x}} \end{bmatrix}, \quad \hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \cdots \\ e_n \end{bmatrix}. \tag{21}$$

Using the method of least squares (see [6]), $\mathbf{e}$ and $\mathbf{X}$ are orthogonal, so

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \tag{22}$$

This equation can be written as

$$\hat{\boldsymbol{\beta}} = \mathbf{S}_{XX}^{-1}\mathbf{S}_{XY} \tag{23}$$

where $\mathbf{S}_{XX}$ and $\mathbf{S}_{XY}$ are the sample covariance matrices

$$\mathbf{S}_{XX} = \frac{1}{n}\mathbf{X}^T\mathbf{X}, \tag{24}$$

$$\mathbf{S}_{XY} = \frac{1}{n}\mathbf{X}^T\mathbf{Y}. \tag{25}$$

3.2. Sampling distribution

On the other hand, we can find the distribution of the estimators:

Property 2. *For model given by equation* (12)*, the least squares estimators for* β_1 *and* β_2 *are given by equation* (23)*. Moreover, the distributions of these estimators is:*

$$\hat{\boldsymbol{\beta}} \sim N\left(\boldsymbol{\beta}, \frac{\sigma^2}{n}\mathbf{S}_{XX}^{-1}\right) \tag{26}$$

The proof of this property can be found in [6].

Property 3. *In the case of random vibrations and large sample size:*

$$\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} \longrightarrow \begin{bmatrix}-\omega_n^2\\ -2\zeta\omega_n\end{bmatrix} \tag{27}$$

$$Var\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} \longrightarrow \frac{\sigma^2}{n}\begin{bmatrix}\frac{2m^2\zeta\omega_n^3}{S_0\pi} & 0\\ 0 & \frac{2m^2\zeta\omega_n}{S_0\pi}\end{bmatrix} \tag{28}$$

Proof. First, it is important to note that (23) can be writen as:

$$\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} = \begin{bmatrix}s_{xx} & s_{x\dot{x}}\\ s_{x\dot{x}} & s_{\dot{x}\dot{x}}\end{bmatrix}^{-1}\begin{bmatrix}s_{x\ddot{x}}\\ s_{\dot{x}\ddot{x}}\end{bmatrix} \tag{29}$$

where s_{ij} are the sample covariance between i and j respectively. In the case of random vibrations ad large sample size, the sample values will converge to the values given in Property 1:

$$s_{xx} \longrightarrow Var[x(t)] = \frac{S_0\pi}{2m^2\zeta\omega_n^3} \tag{30}$$

$$s_{\dot{x}\dot{x}} \longrightarrow \frac{S_0\pi}{2m^2\zeta\omega_n} \tag{31}$$

$$s_{\ddot{x}\ddot{x}} \longrightarrow \frac{S_0\pi\omega}{2m^2\zeta}(1-4\zeta^2) \tag{32}$$

$$s_{x\ddot{x}} \longrightarrow 0 \tag{33}$$

$$s_{x\ddot{x}} \longrightarrow -\frac{S_0\pi}{2m^2\zeta\omega_n} \tag{34}$$

$$s_{\dot{x}\ddot{x}} \longrightarrow -\frac{S_0\pi}{m^2} \tag{35}$$

Therefore:

$$\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} = \begin{bmatrix}s_{xx} & s_{x\dot{x}}\\ s_{x\dot{x}} & s_{\dot{x}\dot{x}}\end{bmatrix}^{-1}\begin{bmatrix}s_{x\ddot{x}}\\ s_{\dot{x}\ddot{x}}\end{bmatrix} \longrightarrow \begin{bmatrix}\frac{S_0\pi}{2m^2\zeta\omega_n^3} & 0\\ 0 & \frac{S_0\pi}{2m^2\zeta\omega_n}\end{bmatrix}^{-1}\begin{bmatrix}-\frac{S_0\pi}{2m^2\zeta\omega_n}\\ -\frac{S_0\pi}{m^2}\end{bmatrix} = \begin{bmatrix}-\omega_n^2\\ -2\zeta\omega_n\end{bmatrix} \tag{36}$$

On the other hand,

$$Var\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} = \frac{\sigma^2}{n}\begin{bmatrix}s_{xx} & s_{x\dot{x}}\\ s_{x\dot{x}} & s_{\dot{x}\dot{x}}\end{bmatrix}^{-1} \longrightarrow \frac{\sigma^2}{n}\begin{bmatrix}\frac{S_0\pi}{2m^2\zeta\omega_n^3} & 0\\ 0 & \frac{S_0\pi}{2m^2\zeta\omega_n}\end{bmatrix}^{-1} = \frac{\sigma^2}{n}\begin{bmatrix}\frac{2m^2\zeta\omega_n^3}{S_0\pi} & 0\\ 0 & \frac{2m^2\zeta\omega_n}{S_0\pi}\end{bmatrix} \tag{37}$$

□

3.3. Delta method

Property 4 (Delta method)**.** *Suppose n random variables* $(X_1, \dots, X_n)$ *with* $(E[X_1], \dots, E[X_n]) = (\mu_1, \dots, \mu_n)$. *Let define* m *new random variables by mean of*

$$Y_1 = g_1(X_1, \dots, X_n) \tag{38}$$
$$\cdots \tag{39}$$
$$Y_m = g_m(X_1, \dots, X_n) \tag{40}$$

And let

$$\nabla g(X_1, \dots, X_n) = \begin{bmatrix}\frac{\partial g_1(X_1,\dots,X_n)}{\partial X_1} & \cdots & \frac{\partial g_1(X_1,\dots,X_n)}{\partial X_n}\\ \cdots & \cdots & \cdots\\ \frac{\partial g_m(X_1,\dots,X_n)}{\partial X_1} & \cdots & \frac{\partial g_m(X_1,\dots,X_n)}{\partial X_n}\end{bmatrix}. \tag{41}$$

Then

$$Var\begin{bmatrix}Y_1\\ \cdots\\ Y_m\end{bmatrix} = \nabla g(\mu_1, \dots, \mu_n) \cdot Var\begin{bmatrix}X_1\\ \cdots\\ X_n\end{bmatrix} \cdot \nabla g(\mu_1, \dots, \mu_n)^T \tag{42}$$

We can use the Delta method to obtain the variance of $\hat{\omega}_n$ and $\hat{\zeta}$ from the variance of $\hat{\beta}_1$ and $\hat{\beta}_2$:

Property 5. *The variance of the estimated natural frequency and damping ratio is given by*

$$Var\begin{bmatrix}\hat{\omega}_n\\ \hat{\zeta}\end{bmatrix} = \frac{m^2\sigma^2}{2nS_0\pi}\begin{bmatrix}\zeta\omega_n & -\zeta^2\\ -\zeta^2 & \frac{(\zeta^2+1)\zeta}{\omega_n}\end{bmatrix} \tag{43}$$

Proof. We use Equation (28) and the Delta method to obtain these variances. In this case we have the parameters estimated with the regression model:

$$(\hat{\beta}_1, \hat{\beta}_2), \text{ with } (E[\hat{\beta}_1], E[\hat{\beta}_2]) = (-\omega_n^2, -2\zeta\omega_n) \tag{44}$$

and we want to find the variance of the estimated modal parameters:

$$(\hat{\omega}_n, \hat{\zeta}) = \left((-\hat{\beta}_1)^{1/2}, -1/2(-\hat{\beta}_1)^{1/2}\hat{\beta}_2\right) \tag{45}$$

Taking derivatives

$$\nabla g(\hat{\beta}_1, \hat{\beta}_2) = \begin{bmatrix} -\frac{1}{2}(-\hat{\beta}_1)^{-1/2} & 0 \\ -\frac{1}{4}(-\hat{\beta}_1)^{-3/2}\hat{\beta}_2 & -\frac{1}{2}(-\hat{\beta}_1)^{-1/2} \end{bmatrix} \tag{46}$$

Evaluating this matrix in the expected values:

$$\nabla g(E[\hat{\beta}_1], E[\hat{\beta}_2]) = \begin{bmatrix} -\frac{1}{2\omega_n} & 0 \\ \frac{\zeta}{2\omega_n^2} & -\frac{1}{2\omega_n} \end{bmatrix} \tag{47}$$

Finally:

$$Var\begin{bmatrix} \hat{\omega}_n \\ \hat{\zeta} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\omega_n} & 0 \\ \frac{\zeta}{2\omega_n^2} & -\frac{1}{2\omega_n} \end{bmatrix} \frac{2m^2\sigma^2}{nS_0\pi} \begin{bmatrix} \zeta\omega_n^3 & 0 \\ 0 & \zeta\omega_n \end{bmatrix} \begin{bmatrix} -\frac{1}{2\omega_n} & 0 \\ \frac{\zeta}{2\omega_n^2} & -\frac{1}{2\omega_n} \end{bmatrix}^T \Rightarrow$$

$$Var\begin{bmatrix} \hat{\omega}_n \\ \hat{\zeta} \end{bmatrix} = \frac{m^2\sigma^2}{2nS_0\pi} \begin{bmatrix} \zeta\omega_n & -\zeta^2 \\ -\zeta^2 & \frac{(\zeta^2+1)\zeta}{\omega_n} \end{bmatrix} \tag{48}$$

□

4. CONCLUSIONS

Acoording to Eq. (43), the main conclussions are:

- The variance of the estimated natural frequencies is small compared to the value of the natural frequency (about the order of the damping ratio):

$$\frac{Var[\hat{\omega}_n]}{\omega_n} \propto \zeta \tag{49}$$

- The variance of the estimated damping ratio is inverse proportional to the natural frequency:

$$\frac{Var[\hat{\zeta}]}{\zeta} \propto \frac{1}{\omega_n} \tag{50}$$

  so the estimation of the damping ratio is better for modes with higher frequency.

- The correlation between the estimated natural frequencies and damping ratios is almost zero (of the order of ζ^2).

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