iomac
International Operational Modal Analysis Conference

11th International Operational Modal Analysis Conference
20 - 23 May 2025 | Rennes, France

Subspace System Identification with Unknown Disturbance Rejection

Qinghua Zhang[1]

[1] Inria, Université Gustave Eiffel, 35042 Rennes Cedex, France. qinghua.zhang@inria.fr

ABSTRACT
System identification usually assumes that the considered system is driven by known inputs and/or stationary random noises. This paper considers the case involving unknown inputs, which are arbitrary disturbances. A typical example is a mechanical structure naturally excited by wind, which changes direction from time to time. To address such disturbances, subspace methods for system identification will incorporate techniques of unknown input observers, which are state estimators with the ability to reject arbitrary unknown disturbances, provided a mathematical model of the system under consideration is available. Simulation results are reported to illustrate the proposed method for system identification while rejecting unknown disturbances.

Keywords: system identification, subspace methods; unknown inputs; disturbance rejection.

1. INTRODUCTION

Subspace system identification is a well developed approach to building linear state-space models from sensor data [1, 2]. In general, the sensor data correspond to the input and the output of the underlying system. In some applications, only the output data are available, not the input. This is the case with operational modal analysis, for which the input corresponds to natural excitations . If the unavailable input is stationary white noises, then some subspace system identification methods, known as Stochastic Subspace Identification (SSI), is applicable [3, 4]. The purpose of this paper is to propose a system identification method to address the case involving unknown inputs, which are arbitrary disturbances. The only assumption on the unknown inputs is that they are restricted to some unknown subspace of the state-space.

In order to avoid confusion between SSI, which processes only output data, and more general subspace system identification handling both input and output data, in this paper the latter will be referred to as

N4SID, which is a short name of *specific numerical algorithms* for subspace state-space system identification [5], though other similar algorithms could also be used for the purpose of this paper.

To address unknown inputs, the design of the new system identification method is inspired by some state estimation methods, which have the ability to reject *arbitrary unknown inputs* [6, 7], also referred to as *disturbances* in this paper. These methods, known as Unknown Input Observers (UIO), require that a mathematical model of the system under consideration is available.

In this paper, the main ideas of subspace system identification and UIO are associated in some way, resulting in an identification procedure rejecting unknown inputs. The ability of UIOs to reject unknown inputs is based on output injection. Usually, state estimation and disturbance rejection are jointly considered in UIO design methods, leading to relatively sophisticated design procedures [6, 8]. Some recent advances simplify UIO design by separating it into two steps: disturbances are first rejected by means of output injection, then a standard state estimation method, typically the Kalman filter is applied, as if there was no unknown disturbance [9]. Based on this simplification, in this paper, disturbance rejection is incorporated into system identification in order to estimate state-space models despite unknown inputs.

The association of system identification and UIO has to overcome some difficulties. First, when disturbance rejection is made with output injection, the injected outputs act as inputs. When a model is built from data by applying a system identification method, the fact that the outputs are injected as inputs may lead to a trivial model. Second, when disturbances are rejected, the information in the state equations corresponding to the subspace of the disturbances is canceled out. As a consequence, part of the state equations cannot be found from data by system identification. These difficulties will be explained in detail in the next sections, together with the proposed solutions.

2. STATE-SPACE FORMULATION AND SUBSPACE SYSTEM IDENTIFICATION

This papar considers state-space systems in the form of

$$x_{k+1} = Ax_k + w_k \tag{1a}$$
$$y_k = Cx_k + v_k \tag{1b}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the vector of sampled sensor measurements, $w_k \in \mathbb{R}^n$ is due to external excitations and modeling uncertainties, $v_k \in \mathbb{R}^m$ is the measurement noises, and $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$ are matrices characterizing the state-space model [10]. Any linear transformation of the state vector x_k leads to another equivalent state-space formulation of the same structure. It means that, viewed from sensor data, the state-space model of a given structure is not unique, and subspace system identification methods build models corresponding to arbitrary linear transformations. Nevertheless, all these equivalent state-space models contain the same modal parameters, notably, *the eigenvalues of the matrix A, which are invariant regarding the linear transformations of* x_k.

When SSI is applied to system (1), it is assumed that w_k is a stationary white noise vector [3, 4]. There are situations where this assumption is by far not true, when w_k is mainly due to external excitations, which may be strongly non-stationary and unknown. Such situations are considered in the next sections.

3. DISTURBANCE REJECTION

To address unknown inputs or disturbances, let us add an extra term Ed_k into the state equation (1a), so that w_k represents only a stationary white noise. Accordingly, the state-space model (1) becomes

$$x_{k+1} = Ax_k + Ed_k + w_k \tag{2a}$$
$$y_k = Cx_k + v_k, \tag{2b}$$

in which $d_k \in \mathbb{R}^p$ is most often referred to as *disturbance*, to avoid possible confusion with the known input or the control variable, and $E \in \mathbb{R}^{n\times p}$ is a matrix specifying the subspace to which the disturbance is restricted. It is assumed that the dimension p of d_k is smaller than the dimension n of x_k. In practice, it is related to the fact that, typically, disturbances affect only part of the degrees of freedom of the underlying system.

The disturbance rejection by output injection [9] was originally studied for the purpose of state estimation, i.e., compute the trajectory of the state vector x_k from sensor measurements y_k, assuming that the system matrices A, C, E are known (as considered in [9], but this assumption will be discarded in the present paper). To recall the main results of [9], let

$$G = E[(CE)^T(CE)]^{-1}(CE)^T. \tag{3}$$

At the discrete time instant $k + 1$, equation (2b) leads to

$$0 = G(y_{k+1} - Cx_{k+1} - v_{k+1}) \tag{4}$$
$$= Gy_{k+1} - GC(Ax_k + Ed_k + w_k) - Gv_{k+1}. \tag{5}$$

Inserting it into (2a) yields

$$x_{k+1} = (I - GC)Ax_k + (I - GC)Ed_k + Gy_{k+1} + (I - GC)w_k - Gv_{k+1}. \tag{6}$$

With G defined in (3), it is straightforward to check that

$$(I - GC)E = 0. \tag{7}$$

Then the unknown disturbance d_k is eliminated from the state equation (6), which, together with (2b), constitutes the transformed state-space model

$$x_{k+1} = (I - GC)Ax_k + Gy_{k+1} + (I - GC)w_k - Gv_{k+1}. \tag{8a}$$
$$y_k = Cx_k + v_k. \tag{8b}$$

The state estimation problem considered in [9] treats Gy_{k+1} as an input term in the state equation. Then the Kalman filter is applied to estimate the trajectory of x_k, assuming that the matrices A, C, E are known. In this paper, *these matrices are unknown*. The purpose is to estimate an equivalent model, while rejecting the unknown disturbance d_k.

4. SUBSPACE IDENTIFICATION WITH DISTURBANCE REJECTION

Subspace system identification methods, such as N4SID [5], are widely applied for building state-space models of the form

$$x_{k+1} = \bar{A}x_k + Bu_k + \bar{w}_k \tag{9a}$$
$$y_k = Cx_k + v_k \tag{9b}$$

by estimating the matrices $\bar{A}, B, C$ form data sequences of u_k and y_k. Usually u_k is the input of the underlying system, y_k is its output. If such a method was directly applied to (8), then it would be related to (9) through

$$\bar{A} = (I - GC)A \tag{10}$$
$$B = G \tag{11}$$
$$u_k = y_{k+1} \tag{12}$$
$$\bar{w}_k = (I - GC)w_k - Gv_{k+1}. \tag{13}$$

Such a direct application cannot work correctly, as it would lead to a *trivial model*, due to the presence of the same sequence y_k both as input and as output. The trivial model would be

$$x_{k+1} = u_k \tag{14a}$$
$$y_k = x_k \tag{14b}$$

corresponding to

$$\bar{A} = 0,\ B = I,\ C = I,\ \bar{w}_k = 0,\ v_k = 0. \tag{15}$$

In order to avoid such trivial results, the idea is to *inject part of y_k corresponding to a subset of the available sensors*. Let a partition of y_k and the corresponding partitions of C and v_k be

$$C = \begin{bmatrix} C^a \\ C^{\bar{a}} \end{bmatrix}, \quad y_k = \begin{bmatrix} y_k^a \\ y_k^{\bar{a}} \end{bmatrix}, \quad v_k = \begin{bmatrix} v_k^a \\ v_k^{\bar{a}} \end{bmatrix}, \tag{16}$$

where the exponent "a" indicates a subset of the rows of C, y_k and v_k, whereas "$\bar{a}$" indicates the complementary rows. The sub-vector y_k^a is then injected instead of the whole vector y_k. Accordingly, the matrix G becomes

$$G^a = E^{(a)}[(C^a E^{(a)})^T (C^a E^{(a)})]^{-1} (C^a E^{(a)})^T. \tag{17}$$

where C^a is part of C as in (16). The matrix E is *not* partitioned. Since E may change due to non-stationary external force excitations, the notation $E^{(a)}$ is used to indicate the matrix E when the data are collected and then processed by injecting y_k^a. Then (9) becomes

$$x_{k+1} = \bar{A}^{(a)} x_k + G^a y_{k+1}^a + \bar{w}_k \tag{18a}$$

$$y_k = C^{(a)} x_k + v_k. \tag{18b}$$

The matrices $\bar{A}^{(a)}$, G^a, $C^{(a)}$ are to be estimated by applying N4SID (or a similar identification method) from sensor data, with $u_k = y_{k+1}^a$ as input and y_k as output. In this result, $C^{(a)}$ is the full matrix C estimated by injecting y_k^a.

Subspace system identification with disturbance rejection assumes that some unknown disturbance d_k is actually present. In practice, if the presence of disturbances (or non-stationary external excitations) is uncertain, the standard SSI should be first tried, without trying to reject any disturbance. If the result is satisfactory by checking the model residul, then it is useless to try the new method with output rejection. Otherwise different partitions of y_k should be tried. Because the matrices $C^a, E^{(a)}$ are unknown, and (17) involves the inverse of $[(C^a E^{(a)})^T (C^a E^{(a)})]$, it is not possible to know in advance if a chosen partition of y_k corresponds to a invertible $[(C^a E^{(a)})^T (C^a E^{(a)})]$.

With an appropriate partition of y_k, N4SID is then applied to (18) with $u_k = y_{k+1}^a$ as input. It yields an estimate of the matrix $\bar{A}^{(a)}$, which is related to the A matrix of (2) in a way similar to (10). More specifically,

$$\bar{A}^{(a)} = (I - G^a C^a) A. \tag{19}$$

It is this matrix $\bar{A}^{(a)}$ which is mainly estimated, together with $G^a, C^{(a)}$ (C^a is part of $C^{(a)}$), by the application of N4SID to (18) with $u_k = y_{k+1}^a$ injected.

It turns out that, with G^a defined in (17), $(I - G^a C^a)$ is a singular (non invertible) square matrix. It is then impossible to deduce A solely from $\bar{A}^a$ through (19), nor its eigenvalues. This is the second difficulty when applying subspace system identification methods with disturbance rejection. Because the arbitrary unknown disturbance d_k cancels out the information in the subspace of the state-space related to $E^{(a)} d_k$, the collected data do not contain sufficient information about A. *This is an intrinsic difficulty caused by arbitrary unknown disturbances, which are typically non-stationary natural excitations*. To overcome this difficulty. different datasets will be combined together so that they compensate each other.

In the example of a mechanical structure naturally excited by wind, the disturbance d_k is in fact due to the non-stationary part of the unknown external wind force. Its non-stationary nature implies that it may affect different degrees of freedom of the underlying mechanical structure, corresponding to different subspaces of the state-space associated to the state-space model (2). In other words, over the time, the matrix E in (2) may change. After the first dataset processed by injecting y_k^a, some time later, a second dataset is collected and then processed by injecting y_k^b, which corresponds to another partition of y_k. Assume that the matrix E is $E^{(b)} \neq E^{(a)}$ when this dataset is collected. Then N4SID is applied, with $u_k = y_{k+1}^b$, to

$$x_{k+1} = \bar{A}^{(b)} x_k + G^b y_{k+1}^b + \bar{w}_k \tag{20a}$$

$$y_k = C^{(b)} x_k + v_k, \tag{20b}$$

yielding estimates of $\bar{A}^{(b)}$, G^b and $C^{(b)}$, similar to the estimates of $\bar{A}^{(a)}$, G^a, $C^{(a)}$ corresponding to (18).

This result is then related to A through an equation similar to (19). Combining the new equation with (19) will help to solve for A. However, there is one more difficulty.

The extra difficulty is related to the fact that the state representation of a linear system is not unique. By applying a linear transformation $\bar{x}_k = T x_k$ to (9) with any square invertible matrix T, an equivalent state-space model of the same system is then obtained with the new state vector $\bar{x}_k$ replacing x_k. Therefore, the same input-output data of (sequences of u_k and y_k) can be generated either with x_k or with $\bar{x}_k$ as state vector. Therefore, the result of system identification is not unique. Nevertheless, after the state transformation $\bar{x}_k = T x_k$, the matrix $\bar{A}$ becomes $T\bar{A}T^{-1}$, which has the same eigenvalues as $\bar{A}$. Therefore, though the result corresponds to $T\bar{A}T^{-1}$ with some unknown matrix T, *the eigenvalues of $\bar{A}$ can still be estimated from this result.*

Due to this non-uniqueness of the result of system identification, after the second identification corresponding to (20) yielding the estimates of $\bar{A}^{(b)}$, G^b and $C^{(b)}$, to link the result with the matrix A, the equation similar to (19) is

$$\bar{A}^{(b)} = (I - G^b C^b) T A T^{-1}, \tag{21}$$

with some unknown matrix T. Moreover, $C^{(a)}$ and $C^{(b)}$ are related by

$$C^{(a)} = C^{(b)} T. \tag{22}$$

The proposed solution is then to extract A from (19), (21) and (22). In these equations, $\bar{A}^{(a)}$, G^a, C^a, $C^{(a)}$ and $\bar{A}^{(b)}$, G^b, C^b, $C^{(b)}$ are already estimated by the two applications of N4SID. The unknowns A and T are then solved for in the least squares sense.

Summary of the whole identification procedure

1. Collect sensor data consisting of a sequence of y_k.
2. Apply standard stochastic subspace identification to the data by assuming a model in the form of (1), in which w_k is a stationary white noise. If the result is satisfactory by checking the model residual, then return the estimated model and stop the procedure, otherwise go to the next step.
3. Choose a subset of sensors corresponding a sub-vector of y_k, say y_k^a. Apply a subspace identification method with the input data $u_k = y_{k+1}^a$ and the output data corresponding to the full vector y_k. If the result is satisfactory by checking the model residual, return the estimates of $\bar{A}^{(a)}$, G^a, C^a, $C^{(a)}$ and go to the next step. Otherwise repeat this step by choosing another sub-vector of y_k.
4. Some time after the collection of the previous data set, collect a new data set when the non-stationary external forces may have significantly changed. For example, if the external forces are due to wind, wait so that the wind direction has changed since the last dataset collection.
5. Choose a subset of sensors corresponding a sub-vector of y_k, say y_k^b. Apply a subspace identification method to the new dataset with the input data $u_k = y_{k+1}^b$ and the output data corresponding to the full vector y_k. If the result is satisfactory by checking the model residual, return the estimates of $\bar{A}^{(b)}$, G^b, G^b, $C^{(b)}$ and go to the next step. Otherwise repeat this step by choosing another sub-vector of y_k. If all the sub-vectors of y_k have already been tried, go back to Step 4.

6. Solve for A and T in the least squares sense the equations

$$\bar{A}^{(a)} = (I - G^a C^a)A \tag{23}$$

$$\bar{A}^{(b)} = (I - G^b C^b)TAT^{-1} \tag{24}$$

$$C^{(a)} = C^{(b)}T. \tag{25}$$

Return the eigenvalues of A and stop the procedure.

5. SIMULATION EXAMPLE

In this preliminary study, consider a simple simulation example to illustrate the proposed identification procedure. Data are generated with a state-space model in the form of (2). Because of the continuous time nature of most physical systems, matrices are first chosen for a continuous time model:

$$A_c = \begin{bmatrix} -0.9 & 0.9 & 0 & 0 \\ -0.9 & 0 & 0.90 & 0 \\ 0 & -0.9 & -0.90 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \; C_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; E_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{26}$$

The corresponding discrete time model matrices, as in (2), are then computed as

$$A = e^{A_c \tau}, \;\; C = C_c, \;\; E = A_c^{-1}(A - I)E_c \tag{27}$$

with the sampling period τ equal to 1 ms. These parameters are used with (2) for generating the first dataset. To simulate the non-stationary nature of external excitation, a second dataset is simulated with

$$E_c = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T, \tag{28}$$

while the other parameters remain unchanged.

The disturbance d_k is simulated as

$$d_k = 5\sin(0.1k) + 5\sin(0.1\sqrt{2}k) + g_k \tag{29}$$

where g_k is a white Gaussian noise of unitary variance, and $\sqrt{2}$ is added so that the deterministic part is (in theory) not periodic. Each component of w_k is independently generated as a white Gaussian noise of variance 0.001, and similarly each component of v_k has the variance 0.0005.

The matrix A has two complex eigenvalues $0.2366 \pm 0.5921i$ and two real eigenvalues 0.4066 and 0.3679. As the mode corresponding to the last eigenvalue is not observable, the data generated by the 4th order system could also be generated by an equivalent 3rd order system. For this reason system identification leads to a 3rd order model.

When system identification with disturbance rejection is applied to the first data set, the agreement between the simulated state sequence x_k and the state sequence $\hat{x}_k$ computed with the estimated model is inspected to evaluate the result. This is possible in a simulation study. Because system identification leads to a model corresponding to an arbitrary basis of the state-space, the linear transformation between x_k and $\hat{x}_k$ is first found by solving a least squares problem. Then the estimated state sequence $\hat{x}_k$ is transformed accordingly to be compared with the simulated state sequence x_k. In Figure 1, $\hat{x}_k$ is the estimated state sequence after the linear state transformation. On the left side, its 3 components are superposed on the 3 corresponding components of x_k. The simulated system has 4 state components, of which the 4th one is not observable. The 3 trajectories shown in Figure 1 are those of the observable components. Because the difference between x_k (blue curves) and $\hat{x}_k$ (red curves) is barely visible, their difference $x_k - \hat{x}_k$ is plotted on the right side of Figure 1.

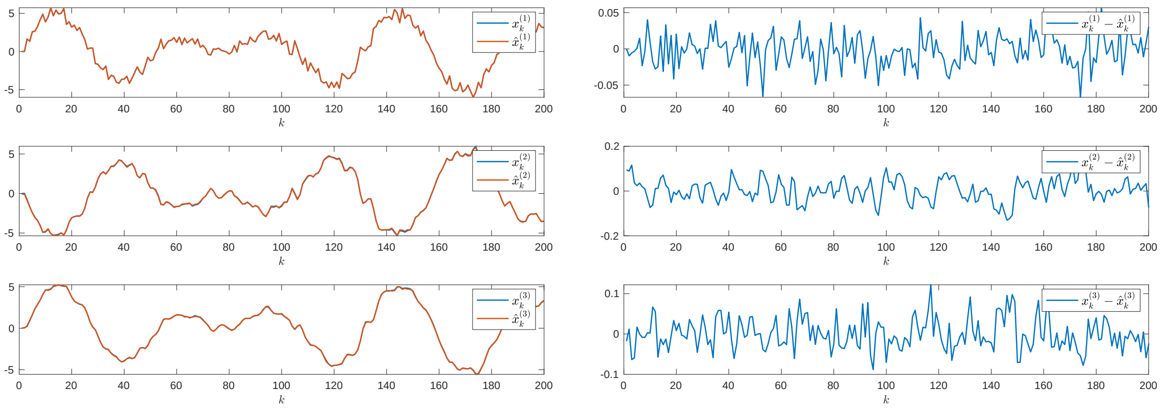

Figure 1: Left: Components of the estimated state sequence $\hat{x}_k$ superposed on the observable components of the simulated state sequence x_k. Right: the difference $x_k - \hat{x}_k$. The estimated state sequence has been linearly transformed so that it corresponds to the same state-space basis as the simulated state sequence.

Then system identification is applied to the second data set, and the matrix A is extracted from the two results of system identification, as summarized in the procedure at the end of the previous section. To compare the estimated complex mode with the simulated complex mode. Let $a \pm bi$ denote the pair of complex eigenvalues of the simulated system, and $\hat{a} \pm \hat{b}i$ their estimates. The relative errors for the real part and the imaginary part are respectively defined as

$$\text{Real part relative error} = \frac{a - \hat{a}}{|a|}, \quad \text{Imaginary part relative error} = \frac{b - \hat{b}}{|b|}. \tag{30}$$

Their mean values and standard deviations evaluated over 100 random realizations of the simulation and estimation are presented in Table 1.

Table 1: Means and standard deviations of the real part relative error and the imaginary part relative error, as defined in (30), evaluated over 100 random realizations.

	Relative error mean	Relative error standard deviation
Real part	-0.0103	0.0467
Imaginary part	-0.0169	0.0366

6. CONCLUSION

In this paper, techniques of subspace system identification and unknown input observers are associated in some way, resulting in a subspace identification method rejecting unknown inputs. A simple simulation exampled has been presented to illustrate the effectiveness of the proposed method for modal properties characterization. Future work will address numerical efficiency in order to apply the method to larger systems.

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