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Wave-based modelling of arbitrarily complex periodic waveguides with nonlinear boundaries

Vincent Mahé [1], *Adrien Mélot* [2], *Benjamin Chouvion*[3], *Christophe Droz*[4]

[1] Université Gustave Eiffel, Inria, COSYS-SII, I4S, France, vincent.mahe@ec-nantes.fr

[2] Université Gustave Eiffel, Inria, COSYS-SII, I4S, France, adrien.melot@inria.fr

[3] Centre de recherche de l'École de l'air et de l'espace, École de l'air et de l'espace, F-13660 Salon-de-Provence, France, benjamin.chouvion@ecole-air.fr

[4] Université Gustave Eiffel, Inria, COSYS-SII, I4S, France, christophe.droz@inria.fr

ABSTRACT

Nonlinear phenomena are becoming increasingly significant in modern mechanical systems due to the transition to lighter, slender structures, and the will to exploit nonlinearities to design more efficient systems. This shift requires accurate system identification techniques to develop reliable models, which are crucial for both the design and monitoring of these systems. However, the identification process is computationally intensive, as numerical models can reach millions of degrees of freedom, with nonlinearities adding significant complexity to the computations. This work aims to address a gap in the current literature related to the simulation of periodic structures with nonlinear boundaries. We propose an efficient computational framework that not only reduces the size of the problem but also provides more physical insights than a classical finite element approach. The proposed method is an extension of the Wave Finite Element Method (WFEM), which combines the Floquet-Bloch theory and the Finite Element Method, to compute the dynamics of periodic waveguides with nonlinear boundaries. The Harmonic Balance Method is used to recast the governing equations into a nonlinear algebraic system, which can then be solved using numerical continuation algorithms. To demonstrate the validity and benefit of this approach, the predictions derived from the proposed methodology are compared to results obtained through conventional Finite Element analysis. Excellent agreement and a notable reduction in computational cost are observed. This methodology enables simulations of realistic models within a feasible timeframe. This aligns with the demands of an R&D product cycle, which requires simulation intensive tools for system identification and digital twinning.

Keywords: model order reduction, periodic waveguides, localised nonlinearities, nonlinear dynamics,

1. INTRODUCTION

The continuously increasing demand for high-performance, smart, lightweight mechanical systems exacerbates the need for advanced numerical models, whose damage status can be assessed and predicted through model-based structural health monitoring and digital twins. These numerical models rely on computationally intensive strategies, whose complexity must be increased further when nonlinearities come into play. Nonlinear effects can emerge from material nonlinearities, coupling between different physics, or large displacements, favoured by the utilisation of slender, lightweight structures. They can also be due to the nature of the contact between two subsystems, with bolted junctions for instance. To keep computation times within acceptable scales, model order reduction techniques are typically used. These consist in reducing the size of the model, typically by expressing the structure's displacement onto a reduced basis. The continuous development of these methods is crucial to open new possibilities in the application of simulation-intensive strategies to realistic structures.

Wave-based model order reduction methods are particularly well-suited for the analysis of waveguides or waveguide assemblies, in which waves travel in a single direction. The basic idea main principle is to express the waveguides' displacement on a wave basis of small dimension, thereby accelerating computation times while keeping an accurate representation of the system's dynamics. Furthermore, waveguide boundaries can be a source of localised nonlinearities, for instance because of the connection between waveguides (*e.g.* bolted junctions), due to the presence of a crack, or because of intentional nonlinearities (*e.g.* nonlinear metamaterials). It is crucial to understand the interplay between a wave and a localised nonlinearity to develop efficient wave-based reduction methods that accurately capture the system's intricate dynamics. The studies discussed thereafter investigate these nonlinear effects on semi-infinite and finite waveguides of simple geometries. In these two cases, the dynamics quantities of interest are the diffusion coefficients and the free and/or forced response, respectively.

Miller [1] and Gaul [2] studied analytically a wave impinging on a dry friction interface and observed that wave diffusion is strongly affected by slipping. Time integration was employed by Ferri [3] to analyse the effect of nonlinear joints on the vibration reduction of a large aerospace structure, noting a hardening behaviour in the forced response along with the emergence of superharmonics and subharmonics. The reflection of a wave travelling in a semi-infinite bar attached to a nonlinear joint was examined by Vakakis and Nayfeh *et al.* [4, 5], who computed nonlinear diffusion relations both analytically and through time integration, revealing the generation of higher wave harmonics. Tang *et al.* [6] extended this work by considering a beam rather than a bar, thereby accounting for evanescent waves. The case of a multi-harmonic incident wave was considered by Abdi *et al.* [7] and demonstrated how the presence of higher harmonics can affect the diffusion of the fundamental one.

The previous approaches are mostly analytical, though some use simple numerical methods like time integration. Though providing an invaluable insight into the system's dynamics, analytical approaches limits the analysis to simple systems, like bars and beams, and to simple, weak nonlinearities. Furthermore, time-domain methods are not efficient to study the frequency content of a system's response. Chouvion [8] proposed a numerical frequency-domain wave-based approach to investigate the dynamic response of a network of bars and beams connected through nonlinear joints. This resulted in a computationally-efficient method allowing to account for strong nonlinearities, as illustrated in the case of a beam connected to an essentially nonlinear oscillator [9]. Balaji *et al.* [10] proposed another wave-based method where analytical wave solutions are used for the linear elements (bars) while the joints are accounted for through nonlinear discrete models. This allows for semi-analytical and numerical simulations of the displacement field in the bar. The key idea of this method, and more generally of wave-based approaches, is to capture the simple physics of the linear domain (waveguide) through wave solutions while granting most of the computation time to the nonlinear boundaries, which is the source of the complex system's response. Though particularly relevant for bar and beam systems, the two aforementioned wave-based approaches are limited to simple waveguide geometries or assemblies thereof.

Dynamic simulations of periodic waveguides of arbitrarily complex geometries, of particular interest in real applications (*e.g.* for system identification or digital twinning), can be performed using the Wave Finite Element Method (WFEM) [11]. This approach takes advantage from the Finite Element Method (FEM) to discretise the waveguide's unit cells while using a Bloch wave basis to express the displacement field, thereby reducing significantly the size of the dynamic problem. The WFEM can be used to compute dispersion curves (and more generally wave modes) [12–17], forced responses [18–21], diffusion coefficients [18, 22], and acoustic emission [14] of periodic waveguide assemblies. It can also be used in the case of 2D periodic structures [23–25]. However, these studies are all limited to linear systems. A first step towards accounting for localised nonlinearities was proposed by Chronopoulos [26] in the purpose of nondestructive testing. The nonlinear source was a crack modelled as a cubic spring linking two semi-infinite composite beams. The super- and subharmonic wave generation was considered using the harmonic balance method. Nevertheless, additional works are required to validate those results, consider multiple Bloch waves propagating simultaneously, propose a formulation adapted to forced responses, couple the WFEM with numerical tools from nonlinear dynamics, and analyse more complex waveguides.

This paper deals with a nonlinear WFEM formulation and its use within frequency-domain methods dedicated to nonlinear simulations, with the aim to allow for computationally-efficient and faithful numerical models of complex structures. Among these nonlinear techniques are i) the harmonic balance method (HBM) [27], which consists in decomposing wave amplitudes on a truncated Fourier basis and projecting the equations of motion on that same basis, ii) a continuation procedure [28], allowing to capture multiple solutions, and iii) the alternating frequency-time (AFT) algorithm [29] to express nonlinear forces in the time domain, where their computation is easier, and then retrieve their harmonic content. The methodology is presented in section 2., followed by an application on a periodic waveguide in section 3. to illustrate the possibilities of the method and its computational efficiency compared to FEM approaches. Conclusions and perspectives are discussed in section 4..

2. METHODOLOGY

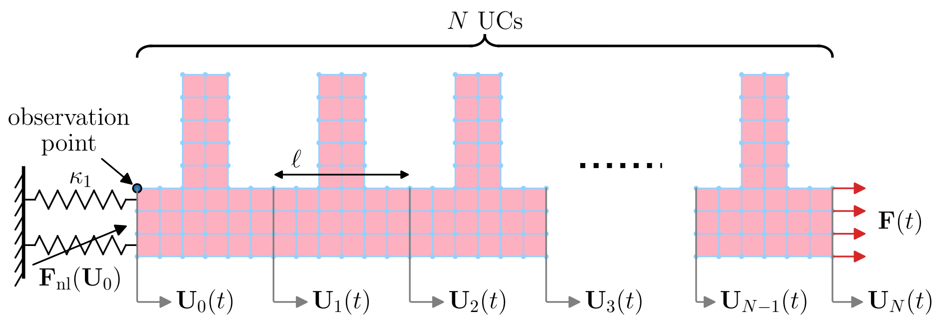

Figure 1: A periodic waveguide made of $N = 100$ unit-cells, attached to a nonlinear joint on its left extremity and subject to a periodic forcing on its right end.

This section details the methodology of the nonlinear WFEM formulation. Figure 1 illustrates a periodic waveguide attached to a nonlinear joint and subject to a periodic forcing. The finite element discretisation of the n^{th} unit-cell (UC) yields the dynamic equation

$$\mathrm{M}_{\mathrm{UC}}\ddot{\mathbf{U}}^{(n)} + \mathrm{C}_{\mathrm{UC}}\dot{\mathbf{U}}^{(n)} + \mathrm{K}_{\mathrm{UC}}\mathbf{U}^{(n)} = \mathbf{F}^{(n)}. \tag{1}$$

$\mathbf{U}^{(n)}$ and $\mathbf{F}^{(n)}$ represent the nodal displacements and forces, respectively, while M_{UC}, C_{UC} and K_{UC} are the mass, viscous damping and stiffness matrices of the UC, respectively. Nodal displacements can be split into interface (index n for the n^{th} interface, starting from 0 up to N) and internal (index I) ones such that

$$\mathbf{U}^{(n)} = \begin{bmatrix} \mathbf{U}_{n-1} \\ \mathbf{U}_I^{(n)} \\ \mathbf{U}_{n+1} \end{bmatrix}. \tag{2}$$

The same goes for the nodal forces. The UC matrices are partitioned accordingly.

The Harmonic Balance Method (HBM) [27] is now used to express the problem in the frequency domain. Time-dependent quantities are decomposed on a Fourier series truncated at harmonic H, and the UC equations are projected onto that same truncated basis. The h^{th} Fourier coefficients of the n^{th} interface displacement are denoted $\mathbf{u}_{n,h}$, such that

$$\mathbf{U}^{(n)}(t) = \Re\left[\sum_{h=0}^{H} \mathbf{u}_{n,h} e^{\mathrm{j}h\omega t}\right], \tag{3}$$

and similar notations are adopted for the forces. Note that the Fourier coefficients of internal degrees-of-freedom (dof) can be expressed as a function of the interface ones, allowing for a dynamic condensation resulting in a system of reduced size $2B$, where B is the number of interface dof.

A projection basis for the Fourier coefficients will now be constructed. To this aim, two adjacent UCs are first connected to establish the equation of motion of an infinite waveguide. Then, due to the $\ell-$periodicity of the waveguide, each Fourier coefficient is sought under the form of a Bloch wave solution such that

$$\mathbf{u}_{n,h} = \lambda_h^n \boldsymbol{\psi}_h. \tag{4}$$

$\lambda_h = e^{-\mathrm{j}k_h \ell}$ is the wave's propagation constant, with k_h the wavenumber, and $\boldsymbol{\psi}_h$ is the waveform. Note that k_h can be complex, in which case $\Re[k_h]$ and $\Im[k_h]$ characterise the propagation and spatial attenuation of the wave, respectively. Substituting Eq. (4) in the infinite waveguide's equation yields a quadratic eigenvalue problem $\mathrm{P}(\lambda_h)\boldsymbol{\psi}_h = \mathbf{0}$ whose $2B$ solutions are split into positive- and negative-going Bloch waves, denoted $(\lambda_{b,h}, \boldsymbol{\psi}_{b,h}^+)$ and $(\lambda_{b,h}^{-1}, \boldsymbol{\psi}_{b,h}^-)$, respectively.

Each Fourier coefficient $h > 0$ can now be projected on the Bloch waves basis. With $q_{b,h}^+$ and $q_{b,h}^-$ the amplitudes of the b^{th} positive- and negative-going waves, one can write the following decomposition of the n^{th} interface displacement

$$\mathbf{U}_n(t) = \mathbf{u}_{n,0} + \Re\left[\sum_{h=1}^{H}\left(\sum_{b=1}^{B} q_{b,h}^+ \lambda_{b,h}^n \boldsymbol{\psi}_{b,h}^+ + q_{b,h}^- \lambda_{b,h}^{N-n} \boldsymbol{\psi}_{b,h}^-\right) e^{\mathrm{j}h\omega t}\right]. \tag{5}$$

The final step in the formulation is to express the boundary conditions through the equations on the 1st and N^{th} UCs, whose forces on the left and right interfaces involve the nonlinear joint force $\mathbf{F}_{\text{nl}}$ and the periodic forcing $\mathbf{F}(t)$. This results in an algebraic system of $2B(1+H)$ equations on the $2B$ static Fourier coefficients of the left and right boundaries and the $2BH$ wave amplitudes, which takes the form

$$\mathrm{Z}\mathbf{x} + \mathbf{f}_{\text{nl}}(\mathbf{x}) - \mathbf{f} = \mathbf{0}, \tag{6}$$

where $\mathbf{x}$ is the vector of unknowns, Z is constructed from the UC matrices and the Bloch modes, while $\mathbf{f}_{\text{nl}}(\mathbf{x})$ and $\mathbf{f}$ contain the projection of the nonlinear forces and periodic forcing on each harmonic, respectively.

Finally, system (6) is solved through a continuation procedure [28], which allows to compute all the solutions (including unstable ones) on a given response branch, even when multiple solutions exist.

This is done by considering the continuation parameter (the forcing frequency in the present case) as an unknown and parametrising all the unknowns by a curvilinear abscissa. The procedure starts with an initial solution, from which a prediction of the next solution is performed using a tangent vector, computed from the Jacobian of system (6). Then, this prediction is corrected along a given path until a given tolerance is met. The path specified is the additional equation, so the system is not underdetermined with the additional unknown ω. An arc is often chosen as the path. It is controlled through a circle equation centred on the initial solution and whose radius is a controllable increment of the curvilinear abscissa. Carrying on this procedure iteratively by considering the corrected solution as a new initial solution constructs the frequency response curve. Delicate steps of this frequency-domain resolution method are the derivation of the Jacobian and the computation of the Fourier coefficients of nonlinear forces. The former involves the derivative of Bloch modes. The later is tackled by computing the nonlinear forces in the time domain and mapping them in the frequency domain using successive inverse discrete Fourier transforms and discrete Fourier transforms [29].

3. APPLICATIONS

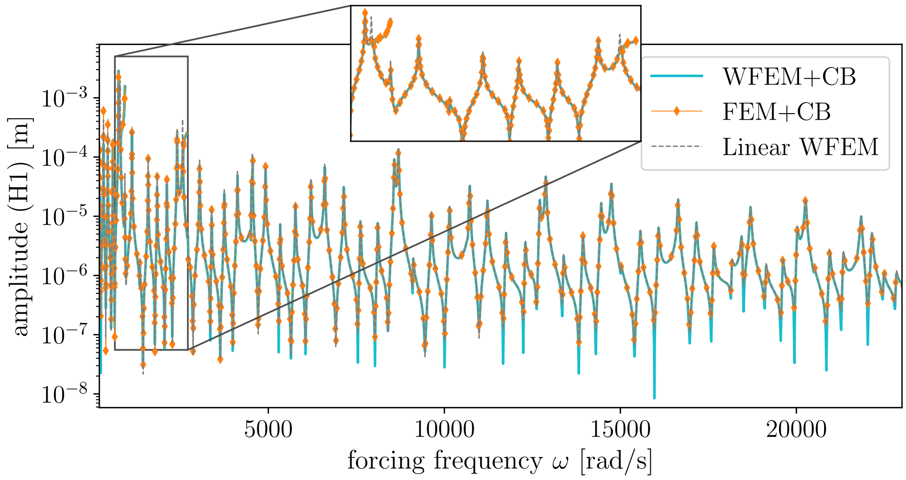

Figure 2: Amplitude response of the 1st harmonic of the vertical displacement at the top left extremity of the waveguide as a function of the forcing frequency. The nonlinear response computed with the nonlinear WFEM is shown as a solid cyan line, while the direct FEM computation is shown as an orange dashed line with diamond markers. Both are superimposed. The linear response is shown in dashed grey. A zoom on the low frequencies allows to better see nonlinear effects (hardening behaviour, with a bending of some resonances towards higher frequencies). The waveguide is meshed by bilinear square elements of size 0.01 [m]. A plane strain assumption is used and the waveguide's constant depth is 0.06 [m]. The material parameters are $E = 200$ [GPa], $\nu = 0.3$, $\rho = 7850$ [kg/m^3], $\eta = 0.2$ [%]. They correspond to the Young's modulus, Poisson's ratio, the density and the structural damping factor, respectively. The stiffness coefficients are $\kappa_1 = 4 \times 10^7$ [N/m] and $\kappa_3 = 1 \times 10^{13}$ [N/m^3]. The forcing amplitude is of 500 [N]. $H = 9$ harmonics are retained in the computations.

The method is applied to the periodic waveguide depicted in Fig. 1 with $n = 100$ UCs, a cubic nonlinearity on the horizontal dof of the left boundary and a harmonic forcing on the right boundary. This resulted in the frequency response curve depicted in Fig. 2, which presents the first harmonic amplitude of the top left dof vertical displacement as a function of the forcing frequency. The nonlinear WFEM formulation (cyan) is compared to a standard FEM resolution (orange), which accounts for the whole waveguide. Both are used conjointly with a Craig-Bampton reduction [30] and with the same HBM, continuation and AFT parameters. The linear response is also shown in dashed grey. Both nonlinear resolution methods match well over the whole frequency band, thereby validating the accuracy of the nonlinear WFEM formulation. Regarding computation times, the WFEM performed 6 times faster than the FEM.

4. CONCLUSIONS

This study presents a wave-based reduction method to compute the dynamic response of periodic waveguides of arbitrarily complex geometry. A unit-cell (UC) is meshed with finite elements and its equations are expressed in the frequency domain using the Harmonic Balance Method (HBM). The Fourier coefficients are then decomposed on a Bloch wave basis, resulting in a nonlinear system on the static Fourier coefficients and Bloch wave amplitudes. The resulting system is solved using a continuation procedure together with an alternating frequency-time method. The accuracy and computational efficiency of this nonlinear WFEM formulation is exposed on a periodic waveguide meshed with square finite elements. Following results will address more advanced systems and investigate particular nonlinear phenomena relative to the bandgap. This original method allows for accurate and computationally efficient simulations of periodic waveguides with localised nonlinearities, thereby facilitating the use of computationally intensive methods such as system identification techniques or digital twinning of complex engineering structures.

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