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Improving Bayesian filters for structural health monitoring applications with subspace-based noise covariance estimates

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ABSTRACT
Bayesian filters are classic engineering tools for estimating the states and the parameters of a dynamic system, often applied in the context of structural health monitoring (SHM) to detect and quantify structural damage. The unmeasured inputs, the modelling errors and the sensor noise give birth to the process and measurement noise that perturb the state and output equations, which affects the state estimation and the subsequent damage quantification. In applications, the noise covariance are generally unknown and their values are oftentimes tuned to optimize user-defined criteria that characterize filtering performance. This can be cumbersome and may not always yield the optimal covariance matrices. Optimization and inversion-based methods have also been developed to estimate these covariances, but such methods are computationally inefficient. The present paper utilizes an existing subspace-based identification method to estimate process and measurement covariances, which are then used in a Bayesian inference-based SHM strategy to estimate states and damage in elements of a structural system. A comparative study is made to determine the effect of using the estimated covariances on the damage detection efficiency and accuracy of the SHM strategy.

Keywords: Kalman filter; noise covariance estimation; structural health monitoring

1. INTRODUCTION

Over the years, Bayesian filters have gained popularity in the field of structural health monitoring (SHM) due to their probabilistic and recursive nature. The dynamics problem is described in the state-space form, where the unobserved states (such as displacement, velocity, etc.) are recursively estimated using the known observed measurements (such as acceleration, etc.). The state-space form has two mathematical models: process equation/model describing the evolution of states in time, and measurement equation describing the relationship between states and measurement outputs [1].

All mathematical models are inherently susceptible to modelling inaccuracies, unknown inputs, and sensor noise, which are addressed as process and measurement noise in the state-space representation, respectively. For linear systems driven by Gaussian noise with known covariance matrices, the Kalman filter (KF) serves as an optimal filter. KF operates in two stages, where, the state probability density function (PDF) is predicted using a system model (prediction) followed by modification of the predicted PDF based on the latest measurement (updation). The updation stage deals with the process and measurement noise, albeit their respective covariances are known to some extent. However, in practice, noise covariance matrices are unknown and usually tuned manually or optimized to uphold KF's performance.

Noise covariance has been estimated using various approaches, eg. Bayesian-based [2], likelihood-based [3], correlation-based approaches [4], etc. Autocovariance least-squares (ALS) [4], has gained prominence in engineering applications as it ensures a unique Kalman predictor gain, enabling optimal state estimation [5]. The high computational efforts in ALS can be resolved using an alternative approach that identified noise covariances from residuals of non-steady-state KF banks. However, these covariances are only determined up to a similarity transform, making them incompatible with model-based KF. This was addressed by a subspace-based noise covariance estimation method [6], which linked system noise covariances to state and output covariances in the filter formulation. Thereby, a unique Kalman gain is obtained, ensuring accurate state estimates for model-based KF.

Noise covariances play an important role in KF, especially in model-based SHM algorithms [7]. The algorithm can under- or over-estimate the damage present in an element or misidentify damage and/or its location if the noise covariances fed to the system are incorrect. Hence, using the afore-mentioned subspace-based noise covariance method, an existing interacting particle-Kalman filter (IPKF)-based SHM method is improved. The effect on damage detection capability of IPKF is studied in the presence/absence of adequate information on the process and measurement noise covariance matrices. The method has been tested on a numerical 6 degree-of-freedom (*dof*) chain of oscillators. Further, noise handling capability of the method has also been studied.

2. BACKGROUND

Consider a discrete-time state-space model of a linear time-invariant mechanical system

$$\begin{aligned} x_k &= Ax_{k-1} + Bu_k + w_k \\ y_k &= Cx_k + Du_k + v_k \end{aligned} \tag{1}$$

where, $x_k \in \mathbb{R}^{n_x}$ and $y_k \in \mathbb{R}^{n_y}$ are the state and output vectors, $u_k \in \mathbb{R}^{n_u}$ is the known input force, $w_k \in \mathbb{R}^{n_x}$ and $v_k \in \mathbb{R}^{n_y}$ are the correlated process and the measurement noise, with the respective covariances Q and R and the cross-covariance S. Matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$ and $D \in \mathbb{R}^{n_y \times n_u}$ are the state-transition, output, observation and the feedthrough matrices, respectively. Hereafter we assume that (A, B, C, D) are known based on a first principle model of the system.

The efficiency of model-driven KF-based SHM approaches is dependent on the adequacy of noise covariance values used, among other factors (quality of model, filter parameters, etc.). Often, insufficient attention is given to the impact of choice of noise covariance matrices used in the implementation of the SHM algorithm, as exhibited by practicing manual tuning, etc. This oversight places an undue burden on the other factors contributing to the efficiency of the SHM approaches, thereby, imposing constraints and reducing overall computational efficiency.

The main idea of this work is to use subspace-based estimates of (Q, R, S) in an IPKF with an application to structural damage assessement. The proposed two-step strategy is as follows: first, the (Q, R, S) triplet is estimated with the subspace-based approach from [6]. The estimates are then used in an IPKF to compute the health parameters associated to each element of the structure. These parameters are encompassed within the vector $\Theta \in \mathbb{R}^{n_m}$, wherein each component's value ranges from 0 to 1. Values 0 and 1 respectively correspond to a 100% damage in the element and 100% healthy state of the element. In this work, Θ corresponds to a reduction in stiffnesses of structural elements.

3. METHODOLOGY

The proposed methodology consists of two parts; the first part involves estimation of the noise covariance matrices and the second comprises parameter estimation. A scheme illustrating the main building blocks of the proposed methodology is depicted in Figure 1.

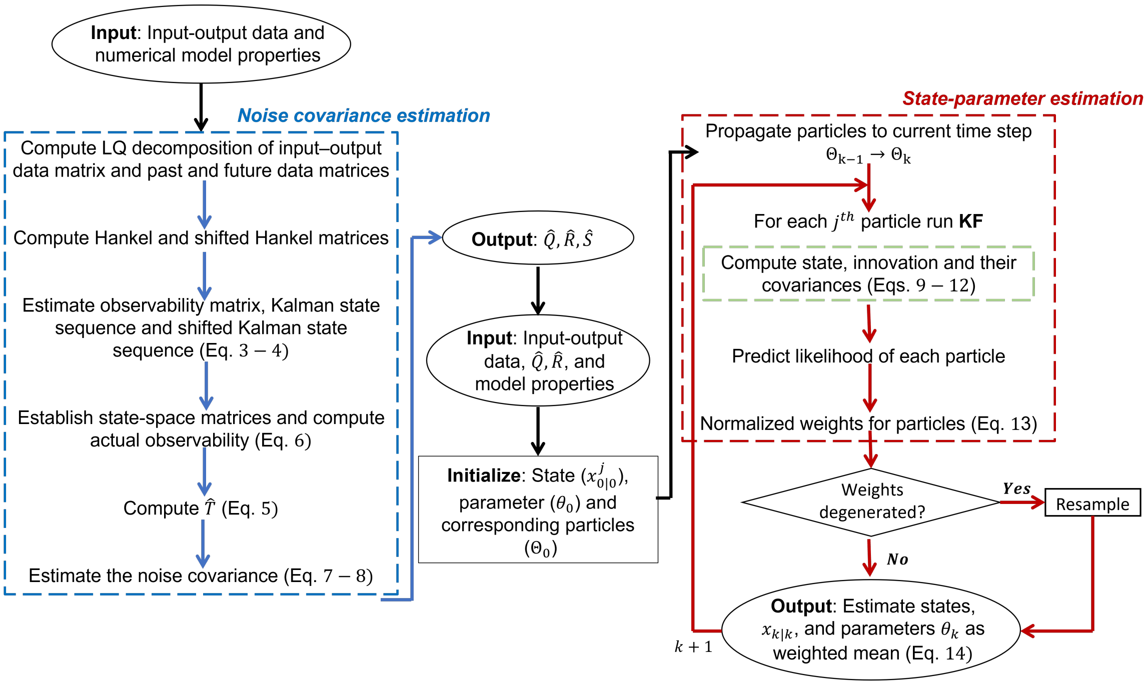

Figure 1: Proposed methodology for damage detection.

3.1. Noise covariance estimation

Hereafter we recap the subspace-based noise covariance estimation approach after [6]. For a discrete signal a_k where k is a time step, define a data matrix

$$\mathcal{A}_{i|j} = \frac{1}{\sqrt{N}} \begin{bmatrix} a_i & a_{i+1} & \dots & a_{i+N-1} \\ a_{i+1} & a_{i+2} & & a_{i+N} \\ \vdots & \vdots & & \vdots \\ a_j & a_{j+1} & \dots & a_{j+N-1} \end{bmatrix} \in \mathbb{R}^{(j-i+1)b \times N}.$$

Subsequently, the 'past' and the 'future' data matrices are defined based on the available u_k and y_k

$$\mathcal{U}^- = \mathcal{U}_{0|p-1},\ \mathcal{U}^+ = \mathcal{U}_{p|2p-1},\ \mathcal{Y}^- = \mathcal{Y}_{0|p-1}, \mathcal{Y}^+ = \mathcal{Y}_{p|2p-1},\ \text{and}\ \mathcal{W}^- = [\mathcal{U}^{-T}\ \mathcal{Y}^{-T}]^T$$
$$\mathcal{Y}^+_- = \mathcal{Y}_{p+1|2p-1},\ \mathcal{U}^+_- = \mathcal{U}_{p+1|2p-1},\ \text{and}\ \mathcal{W}^-_+ = [\mathcal{U}^{-T}\ \mathcal{U}^{pT}\ \mathcal{Y}^{-T}\ \mathcal{Y}^{pT}]^T,$$

where p denotes the data horizon. The estimates of the system states $\hat{\mathcal{X}}^p = \hat{\mathcal{X}}_{p|p}$ and $\hat{\mathcal{X}}^p_+ = \hat{\mathcal{X}}_{p+1|p+1}$ can be obtained from subspaces of an oblique projection of data matrices $\mathcal{Y}^-$, $\mathcal{Y}^+$, $\mathcal{U}^-$, and $\mathcal{U}^+$ using N4SID [8]. To this end, define a Hankel matrix

$$\mathcal{H} = \mathcal{Y}^+/_{\mathcal{U}^+}\mathcal{W}^- = (\mathcal{Y}^+/\mathcal{U}^{+\perp})(\mathcal{W}^-/\mathcal{U}^{+\perp})^{\dagger}\mathcal{W}^-, \tag{2}$$

where, $(\cdot)^{\perp}$ denotes the orthogonal projection of a data matrix, i.e., $\mathcal{A}/\mathcal{B}^{\perp} = \mathcal{A} - \mathcal{A}/\mathcal{B} = \mathcal{A} - \mathcal{A}\mathcal{B}^T(\mathcal{B}\mathcal{B}^T)^{-1}\mathcal{B}$. It can be shown that $\mathcal{H}$ factorizes into the observability matrix Γ and $\hat{\mathcal{X}}^p$ whose estimates can be obtained from Singular Value Decomposition (SVD)

$$\mathcal{H} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \ \hat{\Gamma} = U_1\Sigma_1^{1/2}, \text{ and } \hat{\mathcal{X}}^p = \Sigma_1^{1/2}V_1^T. \tag{3}$$

Note that in practice the model order used in SVD (3) is unknown; a wrong choice will yield a mismatch between the estimate of the observability and its model-driven counterpart, resulting in the estimation of non-physical states. For more details on data-driven model order estimation the interested reader is referred to [9]. A shifted state sequence $\hat{\mathcal{X}}_+^p$ is obtained from the projection of the shifted matrices

$$\hat{\mathcal{X}}_+^p = \hat{\Gamma}_s^{\dagger}\mathcal{H}_s, \tag{4}$$

where $\hat{\Gamma}_s$ is $\hat{\Gamma}$ without the last block row and $\mathcal{H}_s = \mathcal{Y}_-^+/_{\mathcal{U}_-^+}\mathcal{W}_+^-$. State estimates $\hat{\mathcal{X}}^p$ and $\hat{\mathcal{X}}_+^p$ correspond to a different state-space basis than the one defined by the model (A, B, C, D). This can be changed by estimating the change of basis matrix $\hat{T}$ that relates the estimated observability matrix $\hat{\Gamma}$ to the observability matrix Γ, namely

$$\hat{T} = \Gamma^{\dagger}\hat{\Gamma}, \tag{5}$$

where

$$\Gamma = \begin{bmatrix} (C)^T & (CA)^T & (CA^2)^T & \dots & (CA^{p-1})^T \end{bmatrix}^T. \tag{6}$$

Then, the residuals of the estimated model can be obtained based on the shifted state sequences and the model matrices from

$$\begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} = \begin{bmatrix} \hat{T}\hat{\mathcal{X}}_+^p \\ \mathcal{Y}^p \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{T}\hat{\mathcal{X}}^p \\ \mathcal{U}^p \end{bmatrix} \tag{7}$$

and the noise covariance matrices are obtained using

$$\begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{bmatrix} = \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \begin{bmatrix} \rho_w^T & \rho_v^T \end{bmatrix}. \tag{8}$$

3.2. State and parameter estimation

All the health indices are initiated at 1 (healthy damage-free state) following a normal distribution that is applied to obtain corresponding N_p particle vectors, denoted as $\Theta_0 = [\Theta^1, \Theta^2, \dots, \Theta^{N_p}]_{n_m \times N_p}$. Accordingly, N_p different versions of the Θ-dependent systems matrices A_0, C_0, etc., are computed and a corresponding KF is implemented parallely for state estimation.

At the k^{th} time-step, the state estimates and the particles are both updated. Each j^{th} particle is evolved through a random perturbation about its current position, $\theta_k^{j^h} = \alpha\theta_{k-1}^{j^{h^*_{k-1}}} + \mathcal{N}(\delta\theta_{k-1}, \sigma_{k-1}^{\Theta})$ [10], with $1 \leq j \leq N_p$. α is a hyper-parameter, tuning the oscillations in Θ estimation. The propagated particles pass through KF for state evolution, where they have been employed to define the system matrices A_k^j and C_k^j. In addition, each KF yields a likelihood value, computed from its innovation. The particles evolve based on their estimated likelihood depending on the current output and are further re-sampled [11] according to the likelihood values of the corresponding KFs.

Kalman filter. For each j^{th} particle, KF follows the below-mentioned equations for state estimation. Here, Q and R are known from Equation (7).

$$\textit{Prediction}: x_{k|k-1}^j = A_k^j x_{k-1|k-1}^j + Bu_k; \quad P_{k|k-1}^j = A_k^j P_{k-1|k-1}^j {A_k^j}^T + Q \tag{9}$$

$$\textit{Innovation}: \epsilon_k^j = y_k - C_k^j x_{k|k-1}^j - Du_k \tag{10}$$

$$\textit{Kalman gain}: S_k^j = C_k^j P_{k|k-1}^j {C_k^j}^T + R; \quad K_k^j = P_{k|k-1}^j {C_k^j}^T {S_k^j}^{-1} \tag{11}$$

$$\textit{Update}: x_{k|k}^j = x_{k|k-1}^j + K_k^j\epsilon_k^j; \quad P_{k|k}^j = (I - K_k^j C_k^j)P_{k|k-1}^j \tag{12}$$

Particle approximation. With ϵ_k^j and S_k^j, the likelihood of each particle is computed as, $\mathbb{L}(\Theta_k^j) = \frac{1}{\sqrt{|S_k^j|}} \mathrm{e}^{-0.5 \epsilon_k^{jT} S_k^{j^{-1}} \epsilon_k^j}$. The normalized weight for the j^{th} particle is

$$w(\Theta_k^j) = \frac{w(\Theta_k^j)\mathbb{L}(\Theta_k^j)}{\sum_{j=1}^{N_p} w(\Theta_k^j)\mathbb{L}(\Theta_k^j)} \tag{13}$$

Finally, particle approximations for states and parameters are estimated as weighted means,

$$x_{k|k} = \sum_{j=1}^{N_p} w(\Theta_k^j) x_{k|k}^j \quad \text{and} \quad \Theta_k = \sum_{j=1}^{N_p} w(\Theta_k^j) \Theta_k^j \tag{14}$$

4. NUMERICAL STUDY

A numerical study has been conducted on a 6 degrees of freedom (*dof*) spring-mass-damper (k-m-c) system (cf. Figure 1) to depict the relevance of estimating noise covariances for model-driven damage detection SHM algorithms. Each element of the considered k-m-c system is identical, where stiffness, k = 200 N, and mass, m = 0.05 kg. Further, classic mass-proportional damping has been considered, with modal damping ratio equals 0.02.

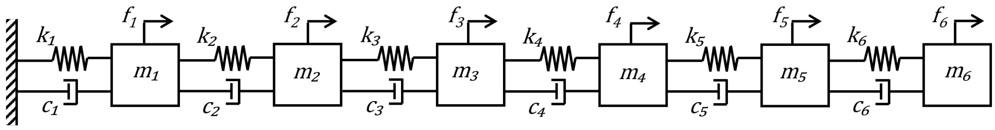

Figure 2: 6-*dof* spring-mass-damper system.

The numerical model is simulated at a rate of 50 Hz for 10e4 sample points, with external force modelled as white Gaussian noise (WGN) acting on all *dof*s during the entire simulation. While the entire data is used to determine noise covariance matrices, only 2048 sample points are used to determine damage in the structure due to prompt detection in most of the cases. The output and input channels are known at all *dof*s. 1000 particles have been used for PF, while $\alpha = 0.90$ which provides a good balance between stability and adaptability in particle evolution (lower values (< 0.8) result in slow but stable convergence while higher values (> 0.95) emphasize new estimates for faster updates, risking instability [7]).

4.1. Effect of choice of noise covariance matrix

A comparative study has been conducted to investigate the effect of noise covariance matrices originating from various sources on the damage detection capability of the IPKF. Following cases have been considered:

1. Actual measurement noise covariance matrix is used in IPKF for state and parameter estimation.

2. Measurement noise covariance matrix obtained from first step (subspace covariance estimation, $\hat{R}$) of the proposed methodology has been used in IPKF (second step of proposed methodology).

3. Many a times with noise covariance matrices can be manually tuned for Bayesian filtering-based SHM algorithms while benchmarking. This may result in improper selection of R for SHM. This is mimicked by supplying $2 \times \hat{R}$ to the IPKF (computed as second step of proposed methodology).

4. The covariance of measurement data is computed from '*cov*' function of MATLAB and directly used in IPKF.

Meanwhile the other covariance matrices are kept the same. Both output and input signals from the simulation have been contaminated with WGN, such that the signal to noise ratio (SNR) is 40 dB (considered as good signal) for all signals. A 50% damage is inflicted on the second element throughout the simulation.

Case 1 provides the most accurate Θ estimates (cf. Figure 4) resulting from better estimated measurements (cf. Figure 3). Figure 3 compares the measurement estimated for different cases. Performance of IPKF in terms of state/measurement estimation for the afore-mentioned cases, from least to most mean relative error can be ranked as $1 > 2 > 3 > 4$.

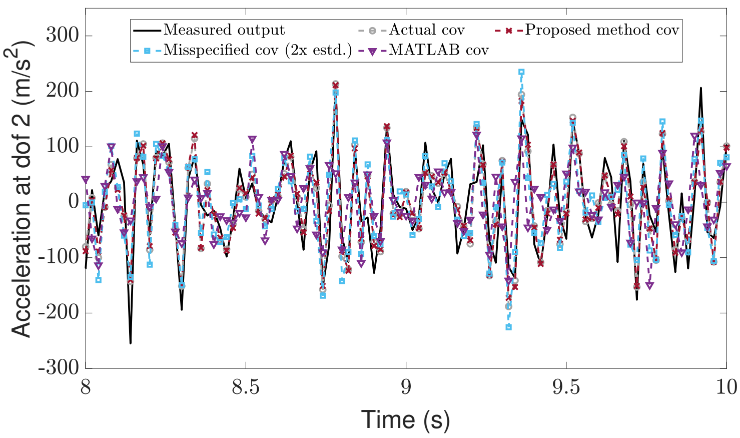

Figure 3: Measurement estimations by IPKF for different noise covariance matrix values; *where*, cov = covariance, estd. = estimated measurement noise covariance matrix - proposed methodology.

Following a similar trend observed in measurement estimation, the health indices Θ also provide similar inference. Figure 4 has a pair of plots, distinguished by color, corresponding to each case (1-4). The solid lines/plots correspond to θ of 2^{nd} element where damage was inflicted during simulation. The dashed line corresponds to the θs of elements with no damage. θ corresponding to 3^{rd} element has been plotted to represent the undamage elements, for better readability of the plot. As expected, the Θ estimation is most prompt and accurate for cases 1 and 2, where the actual and the compatible (proposed method) R has been used. Meanwhile, for cases 3 and 4, the Θ estimation has a slightly reduced but acceptable accuracy, with a delayed detection in damage. This may hinder real-time SHM applications. Θ estimation in both cases 3 and 4 when compared to that of case 2, highlights the effectiveness of estimating noise covariance matrices that are compatible to the model-driven KF-based SHM methods. It should be noted that as the assumed noise covariance noise matrix (case 3) strays away from $\hat{R}$, Θ estimation degrades further, with no distinct damage observed for $R = 10\hat{R}$.

4.2. SNR study

The performance of the proposed methodology in terms of its handling of signals contaminated by different SNR levels is studied next. Both output and input signals are contaminated with different noise levels. The damage is inflicted at 2^{nd} *dof* by reducing its stiffness to 50%.

The results obtained are shown in Table 1. The columns correspond to change in SNR in force signals, and rows show change in SNR in measurement signals. Performance is based on two criteria: whether the location of damage is correct ($\checkmark/\times$) and the accuracy of quantification. Accuracy ($\%$) is measured as the ratio of mean of θ for last 1000 iterations versus actual θ value corresponding to the damage. For the present example, it is observed that the proposed methodology is capable of detecting damage under different SNRs. It is also observed lower SNR in measurement affect the accuracy of detection more than a lower SNR in input signals. It should be noted KF-based SHM approaches are better equipped to deal with input signals represented by WGN. No false positive is observed for any case.

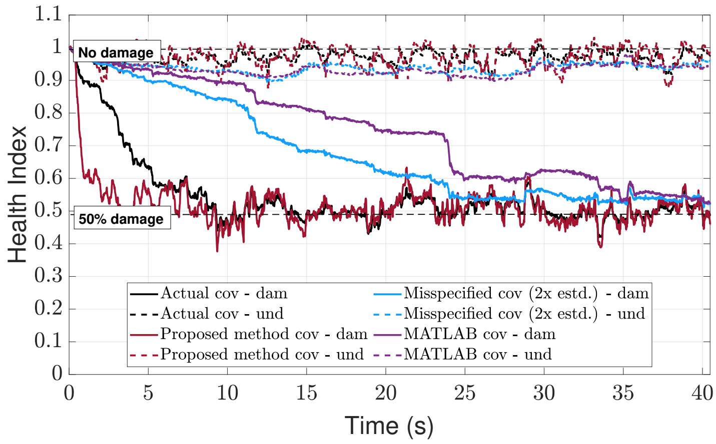

Figure 4: Health index estimation by IPKF for various noise covariance matrix values; *where*, cov = covariance, estd. = estimated measurement noise covariance matrix - proposed methodology, dam = damage, und = no damage.

Table 1: Signal-noise-ratio (SNR) study. ✓ / × = damage detected / not detected - Accurracy of detection in %.

	SNR in input force data (dB)			
SNR in output data (dB)	10	20	40	60
10	✓ - 91.01	✓ - 92.33	✓ - 93.18	✓ - 92.98
20	✓ - 92.99	✓ - 94.05	✓ - 93.60	✓ - 94.40
40	✓ - 94.04	✓ - 95.24	✓ - 95.12	✓ - 95.69
60	✓ - 97.93	✓ - 97.88	✓ - 98.15	✓ - 98.75

4.3. Output only IPKF

The results from the SNR study, prompted an investigation on the efficiency of IPKF under different types of forces, when no input data is provided while same noise covariance matrices are provided on each occasion. The system is simulated under different types of force inputs, stationary noise, periodic, and non-stationary inputs (cf. Table 2). Damage is kept the same as previous studies - 50% at 2^{nd} *dof*. Only the measurement data from these simulations are supplied to IPKF while the force is treated unknown (eliminating u_k from Equations (9) and (10)). The noise covariance matrices are kept the same as obtained from case 2 in Section 4.1..

A decrease in accuracy of the Θ estimation is observed (cf. 1^{st} column in Table 2), when compared to the accuracy seen in case 2 (Section 4.1.) where the inputs were known. Damage has been correctly localized for each of the cases with acceptable accuracy in quantification for most cases ($\geq$ 90%). As the noise covariance characteristics of the new measurements become distant to the supplied matrices, the decrease in accuracy is well demonstrated (second to last column in Table 2). However, for all the unknown input type cases, damage location is distinctly identified with no false alarms.

Table 2: Performance of output-only IPKF supplied with same noise covariance matrices

Stationary noise input			Periodic input				Non-stationary input
$\mathcal{N}(0,1)$	$\mathcal{N}(10,1)$	$\mathcal{N}(10,4)$	$2sin(10t)$	$10sin(10t)$	$2sin(100t)$	$10sin(100t)$	
✓ - 91.30	✓ - 90.47	✓ - 86.89	✓ - 91.62	✓ - 90.57	✓ - 86.17	✓ - 56.16	✓ - 88.49

5. CONCLUSIONS

A two-step methodology to improve SHM approaches has been presented. The first step estimates the process and meeasurement noise covariance matrices using subspace based identification technique, which are provided to IPKF (second step) to assist in improved damage detection. The study emphasizes how adequate representation of noise in signals in terms of noise covariance matrices is pertinent and provides better support to model-driven KF-based SHM algorithms.

The subspace-based noise covariance estimation assumes the system to be linear time invariant (LTI) resulting in estimation of constant noise covariance matrices within the sample time. This limits the method to detect changes in the system during damage. Some leverage is provided in terms of signal noise ratio, where good performance of the method is observed even for low signal strengths of both inputs and outputs. The methodology applies for both input-output and output-only noise covariance estimation and IPKF approaches.

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